FOCUS ON THEORY

SEPARATION OF VARIABLES

We have seen how to sketch solution curves of a differential equation using a slope field. Now we see how to solve certain differential equations analytically, finding an equation for the solution curve.

First, we look at a familiar example, the differential equation

$$\frac{dy}{dx} = -\frac{x}{y},$$

whose solution curves are the circles

$$x^2 + y^2 = C.$$

We can check that these circles are solutions by differentiation; the question now is how they were obtained. The method of *separation of variables* works by putting all the xs on one side of the equation and all the ys on the other, giving

$$y\,dy = -x\,dx.$$

We then integrate each side separately:

$$\int y\,dy = -\int x\,dx,$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + k.$$

This gives the circles we were expecting:

$$x^2 + y^2 = C \qquad \text{where } C = 2k.$$

You might worry about whether it is legitimate to separate the dx and the dy. The reason it can be done is explained at the end of this section.

The Exponential Growth and Decay Equations

We use separation of variables to derive the general solution of the equation

$$\frac{dy}{dt} = ky.$$

Separating variables, we have

$$\frac{1}{y}dy = k\,dt,$$

and integrating,

$$\int \frac{1}{y}\,dy = \int k\,dt,$$

gives

$$\ln|y| = kt + C \quad \text{for some constant } C.$$

Solving for $|y|$ leads to

$$|y| = e^{kt+C} = e^{kt}e^{C} = Ae^{kt}$$

where $A = e^C$, so A is positive. Thus,

$$y = (\pm A)e^{kt} = Be^{kt}$$

where $B = \pm A$, so B is any nonzero constant. Even though there's no C leading to $B = 0$, we can have $B = 0$ because $y = 0$ is a solution to the differential equation. We lost this solution when we divided through by y at the first step. Thus we have derived the solution used earlier in the chapter:

$$y = Be^{kt} \quad \text{for any constant } B.$$

Example 1 Find all solutions of

$$\frac{dy}{dt} = k(y - A).$$

Solution We separate variables and integrate:

$$\int \frac{1}{y-A}\, dy = \int k\, dt.$$

This gives

$$\ln|y - A| = kt + D,$$

where D is a constant of integration. Solving for y leads to

$$|y - A| = e^{kt+D} = e^{kt}e^{D} = Be^{kt}$$

or

$$y - A = (\pm B)e^{kt} = Ce^{kt}$$
$$y = A + Ce^{kt}.$$

Also, $C = 0$ gives a solution. This is the same result we used earlier.

Example 2 Find and sketch the solution to

$$\frac{dP}{dt} = 2P - 2Pt \qquad \text{satisfying } P = 5 \text{ when } t = 0.$$

Solution Factoring the right-hand side gives

$$\frac{dP}{dt} = P(2 - 2t).$$

Separating variables, we get

$$\int \frac{dP}{P} = \int (2 - 2t)\, dt,$$

so

$$\ln|P| = 2t - t^2 + C.$$

Solving for P leads to

$$|P| = e^{2t-t^2+C} = e^{C}e^{2t-t^2} = Ae^{2t-t^2}$$

with $A = e^C$, so $A > 0$. In addition, $A = 0$ gives a solution. Thus the general solution to the differential equation is

$$P = Be^{2t-t^2} \quad \text{for any } B.$$

To find the value of B, substitute $P = 5$ and $t = 0$ into the general solution, giving

$$5 = Be^{2\cdot 0 - 0^2} = B$$

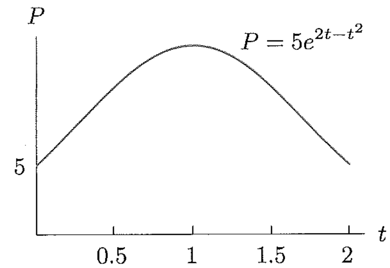

Figure 10.52: Bell-shaped solution curve

so

$$P = 5e^{2t-t^2}.$$

The graph of this function is in Figure 10.52. Since the solution can be rewritten as

$$P = 5e^{1-1+2t-t^2} = 5e^1 e^{-1+2t-t^2} = (5e)e^{-(t-1)^2},$$

the graph has the same shape as the graph of $y = e^{-t^2}$, the bell-shaped curve of statistics. Here the maximum, normally at $t = 0$, is shifted one unit to the right to $t = 1$.

Justification for Separation of Variables

Suppose a differential equation can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

Provided $f(y) \neq 0$, we write $f(y) = 1/h(y)$ so the right-hand side can be thought of as a fraction,

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

If we multiply through by $h(y)$ we get

$$h(y)\frac{dy}{dx} = g(x).$$

Thinking of y as a function of x, so $y = y(x)$, and $dy/dx = y'(x)$, we can rewrite the equation as

$$h(y(x)) \cdot y'(x) = g(x).$$

Now integrate both sides with respect to x:

$$\int h(y(x)) \cdot y'(x)\, dx = \int g(x)\, dx.$$

The form of the integral on the left suggests that we use the substitution $y = y(x)$. Since $dy = y'(x)\, dx$, we get

$$\int h(y)\, dy = \int g(x)\, dx.$$

If we can find antiderivatives of h and g, then this gives the equation of the solution curve.

Note that transforming the original differential equation,

$$\frac{dy}{dx} = \frac{g(x)}{h(y)},$$

into

$$\int h(y)\,dy = \int g(x)\,dx$$

looks as though we have treated dy/dx as a fraction, cross-multiplied and then integrated. Although that's not exactly what we have done, you may find this a helpful way of remembering the method. In fact, the dy/dx notation was introduced by Leibniz to allow shortcuts like this (more specifically, to make the chain rule look like cancellation).

Problems on Separation of Variables

Use separation of variables to find the solutions to the differential equations in Problems 1–12, subject to the given initial conditions.

1. $\dfrac{dP}{dt} = -2P, \quad P(0) = 1$

2. $\dfrac{dL}{dp} = \dfrac{L}{2}, \quad L(0) = 100$

3. $P\dfrac{dP}{dt} = 1, \quad P(0) = 1$

4. $\dfrac{dm}{ds} = m, \quad m(1) = 2$

5. $2\dfrac{du}{dt} = u^2, \quad u(0) = 1$

6. $\dfrac{dz}{dy} = zy, \quad z = 1$ when $y = 0$

7. $\dfrac{dR}{dy} + R = 1, \quad R(1) = 0.1$

8. $\dfrac{dy}{dt} = \dfrac{y}{3+t}, \quad y(0) = 1$

9. $\dfrac{dz}{dt} = te^z$, through the origin

10. $\dfrac{dy}{dx} = \dfrac{5y}{x}, \quad y = 3$ where $x = 1$

11. $\dfrac{dy}{dt} = y^2(1+t), \quad y = 2$ when $t = 1$

12. $\dfrac{dz}{dt} = z + zt^2, \quad z = 5$ when $t = 0$

13. Determine which of the following differential equations is separable. Do not solve the equations.

(a) $y' = y$ **(b)** $y' = x + y$
(c) $y' = xy$ **(d)** $y' = \sin(x+y)$
(e) $y' - xy = 0$ **(f)** $y' = y/x$
(g) $y' = \ln(xy)$ **(h)** $y' = (\sin x)(\cos y)$
(i) $y' = (\sin x)(\cos xy)$ **(j)** $y' = x/y$
(k) $y' = 2x$ **(l)** $y' = (x+y)/(x+2y)$

Use separation of variables to solve the differential equations in Problems 14–19. Assume a, b, and k are nonzero constants.

14. $\dfrac{dP}{dt} = P - a$

15. $\dfrac{dQ}{dt} = b - Q$

16. $\dfrac{dP}{dt} = k(P - a)$

17. $\dfrac{dR}{dt} = aR + b$

18. $\dfrac{dP}{dt} - aP = b$

19. $\dfrac{dy}{dt} = ky^2(1+t^2)$

20. **(a)** Find the general solution to the differential equation modeling how a person learns:

$$\frac{dy}{dt} = 100 - y.$$

(b) Plot the slope field of this differential equation and sketch solutions with $y(0) = 25$ and $y(0) = 110$.
(c) For each of the initial conditions in part (b), find the particular solution and add to your sketch.
(d) Which of these two particular solutions could represent how a person learns?

21. **(a)** Sketch the slope field for the differential equation $dy/dx = xy$.
(b) Sketch several solution curves.
(c) Solve the differential equation analytically.