FOCUS ON THEORY

THEOREMS ABOUT DEFINITE INTEGRALS

The Second Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus tells us that if we have a function F whose derivative is a continuous function f, then the definite integral of f is given by

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

We now take a different point of view. If a is fixed and the upper limit is x, then the value of the integral is a function of x. We define a new function G on the interval by

$$G(x) = \int_a^x f(t)\,dt.$$

To visualize G, suppose that f is positive and $x > a$. Then $G(x)$ is the area under the graph of f in Figure 5.78. If f is continuous on an interval containing a, then it can be shown that G is defined for all x on that interval.

We now consider the derivative of G. Using the definition of the derivative,

$$G'(x) = \lim_{h \to 0} \frac{G(x+h) - G(x)}{h}.$$

Suppose f and h are positive. Then we can visualize

$$G(x) = \int_a^x f(t)\,dt$$

and

$$G(x+h) = \int_a^{x+h} f(t)\,dt$$

as areas, which leads to representing

$$G(x+h) - G(x) = \int_x^{x+h} f(t)\,dt$$

as a difference of two areas.

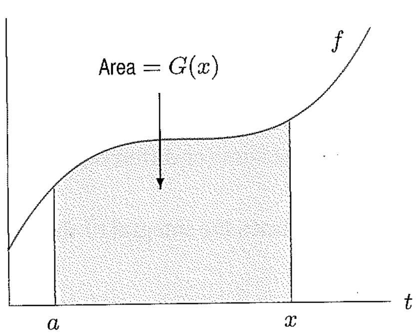

Figure 5.78: Representing $G(x)$ as an area

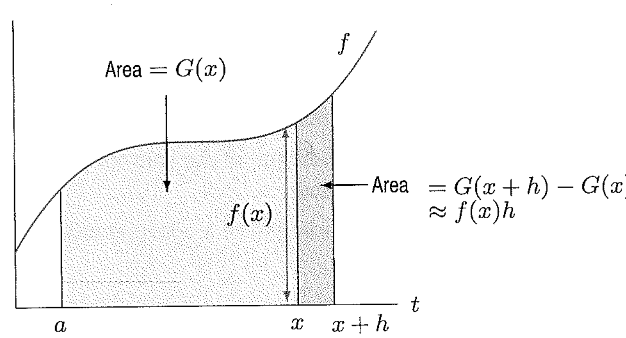

Figure 5.79: $G(x+h) - G(x)$ is the area of a roughly rectangular region

From Figure 5.79, we see that, if h is small, $G(x+h)-G(x)$ is roughly the area of a rectangle of height $f(x)$ and width h (shaded darker in Figure 5.79), so we have

$$G(x+h)-G(x)\approx f(x)h,$$

hence

$$\frac{G(x+h)-G(x)}{h}\approx f(x).$$

The same result holds when h is negative, suggesting that

$$G'(x)=\lim_{h\to 0}\frac{G(x+h)-G(x)}{h}=f(x).$$

This result is another form of the Fundamental Theorem of Calculus. It is usually stated as follows:

Second Fundamental Theorem of Calculus

If f is a continuous function on an interval, and if a is any number in that interval, then the function G defined on the interval by

$$G(x)=\int_a^x f(t)\,dt$$

has derivative f; that is, $G'(x)=f(x)$.

Properties of the Definite Integral

In this chapter, we have used the following properties to break up definite integrals.

Sums and Multiples of Definite Integrals

If a, b, and c are any numbers and f and g are continuous functions, then

1. $\displaystyle\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\int_a^b f(x)\,dx.$
2. $\displaystyle\int_a^b (f(x)\pm g(x))\,dx=\int_a^b f(x)\,dx\pm\int_a^b g(x)\,dx.$
3. $\displaystyle\int_a^b cf(x)\,dx=c\int_a^b f(x)\,dx.$

In words:

1. The integral from a to c plus the integral from c to b is the integral from a to b.
2. The integral of the sum (or difference) of two functions is the sum (or difference) of their integrals.
3. The integral of a constant times a function is that constant times the integral of the function.

These properties can best be visualized by thinking of the integrals as areas or as the limit of the sum of areas of rectangles.