FOCUS ON THEORY

ESTABLISHING THE DERIVATIVE FORMULAS

The graph of $f(x) = x^2$ suggests that the derivative of x^2 is $f'(x) = 2x$. However, as we saw in the Focus on Theory section in Chapter 2, to be sure that this formula is correct, we have to use the definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

As in Chapter 2, we simplify the difference quotient and then take the limit as h approaches zero.

Example 1 Confirm that the derivative of $g(x) = x^3$ is $g'(x) = 3x^2$.

Solution Using the definition, we calculate $g'(x)$:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

Multiplying out $\longrightarrow$ $$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

Simplifying $\longrightarrow$ $$= \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2.$$

Looking at what happens as $h \to 0$

So $g'(x) = \frac{d}{dx}(x^3) = 3x^2$.

Example 2 Give an informal justification that the derivative of $f(x) = e^x$ is $f'(x) = e^x$.

Solution Using $f(x) = e^x$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right).$$

What is the limit of $\frac{e^h - 1}{h}$ as $h \to 0$? The graph of $\frac{e^h - 1}{h}$ in Figure 3.27 suggests that $\frac{e^h - 1}{h}$ approaches 1 as $h \to 0$. In fact, it can be proved that the limit equals 1, so

$$f'(x) = \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h}\right) = e^x \cdot 1 = e^x.$$

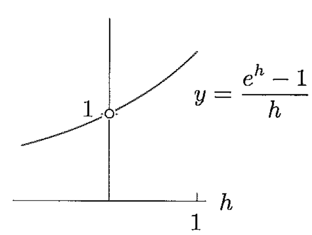

Figure 3.27: What is $\lim_{h\to 0} \frac{e^h-1}{h}$?

Example 3 Show that if $f(x) = 2x^2 + 1$, then $f'(x) = 4x$.

Solution We use the definition of the derivative with $f(x) = 2x^2 + 1$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(2(x+h)^2 + 1) - (2x^2 + 1)}{h}$$
$$= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) + 1 - 2x^2 - 1}{h} = \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + 1 - 2x^2 - 1}{h}$$
$$= \lim_{h\to 0} \frac{4xh + 2h^2}{h} = \lim_{h\to 0} \frac{h(4x + 2h)}{h}$$

To find the limit, look at what happens when h is close to 0, but $h \neq 0$. Simplifying, we have

$$f'(x) = \lim_{h\to 0} \frac{h(4x + 2h)}{h} = \lim_{h\to 0} (4x + 2h) = 4x$$

because as h gets close to 0, we know that $4x + 2h$ gets close to $4x$.

Using the Chain Rule to Establish Derivative Formulas

We use the chain rule to justify the formulas for derivatives of $\ln x$ and of a^x.

Derivative of $\ln x$

We'll differentiate an identity that involves $\ln x$. In Section 1.6, we have $e^{\ln x} = x$. Differentiating gives

$$\frac{d}{dx}(e^{\ln x}) = \frac{d}{dx}(x) = 1.$$

On the left side, since e^x is the outside function and $\ln x$ is the inside function, the chain rule gives

$$\frac{d}{dx}(e^{\ln x}) = e^{\ln x} \cdot \frac{d}{dx}(\ln x).$$

Thus, as we said in Sections 3.2,

$$\frac{d}{dx}(\ln x) = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Derivative of a^x

Graphical arguments suggest that the derivative of a^x is proportional to a^x. Now we show that the constant of proportionality is $\ln a$. For $a > 0$, we use the identity from Section 1.6:

$$\ln(a^x) = x \ln a.$$

On the left side, using $\frac{d}{dx}(\ln x) = \frac{1}{x}$ and the chain rule gives

$$\frac{d}{dx}(\ln a^x) = \frac{1}{a^x} \cdot \frac{d}{dx}(a^x).$$

Since $\ln a$ is a constant, differentiating the right side gives

$$\frac{d}{dx}(x \ln a) = \ln a.$$

Since the two sides are equal, we have

$$\frac{1}{a^x} \frac{d}{dx}(a^x) = \ln a.$$

Solving for $\frac{d}{dx}(a^x)$ gives the result of Section 3.2. For $a > 0$,

$$\frac{d}{dx}(a^x) = (\ln a)a^x.$$

The Product Rule

Suppose we want to calculate the derivative of the product of differentiable functions, $f(x)g(x)$, using the definition of the derivative. Notice that in the second step below, we are adding and subtracting the same quantity: $f(x)g(x+h)$.

$$\begin{aligned}
\frac{d[f(x)g(x)]}{dx} &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h}\right]
\end{aligned}$$

Taking the limit as $h \to 0$ gives the product rule:

$$(f(x)g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

The Quotient Rule

Let $Q(x) = f(x)/g(x)$ be the quotient of differentiable functions. Assuming that $Q(x)$ is differentiable, we can use the product rule on $f(x) = Q(x)g(x)$:

$$f'(x) = Q'(x)g(x) + Q(x)g'(x).$$

Substituting for $Q(x)$ gives

$$f'(x) = Q'(x)g(x) + \frac{f(x)}{g(x)}g'(x).$$

Solving for $Q'(x)$ gives

$$Q'(x) = \frac{f'(x) - \dfrac{f(x)}{g(x)}g'(x)}{g(x)}.$$

Multiplying the top and bottom by $g(x)$ to simplify gives the quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Problems on Establishing the Derivative Formulas

For Problems 1–7, use the definition of the derivative to obtain the following results.

1. If $f(x) = 2x + 1$, then $f'(x) = 2$.
2. If $f(x) = 5x^2$, then $f'(x) = 10x$.
3. If $f(x) = 2x^2 + 3$, then $f'(x) = 4x$.
4. If $f(x) = x^2 + x$, then $f'(x) = 2x + 1$.
5. If $f(x) = 4x^2 + 1$, then $f'(x) = 8x$.
6. If $f(x) = x^4$, then $f'(x) = 4x^3$. [Hint: $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.]
7. If $f(x) = x^5$, then $f'(x) = 5x^4$. [Hint: $(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$.]
8. **(a)** Use a graph of $g(h) = \dfrac{2^h - 1}{h}$ to explain why we believe that $\displaystyle\lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.6931$.

   (b) Use the definition of the derivative and the result from part (a) to explain why, if $f(x) = 2^x$, we believe that $f'(x) \approx (0.6931)2^x$.
9. Use the definition of the derivative to show that if $f(x) = C$, where C is a constant, then $f'(x) = 0$.
10. Use the definition of the derivative to show that if $f(x) = b + mx$, for constants m and b, then $f'(x) = m$.
11. Use the definition of the derivative to show that if $f(x) = k \cdot u(x)$, where k is a constant and $u(x)$ is a function, then $f'(x) = k \cdot u'(x)$.
12. Use the definition of the derivative to show that if $f(x) = u(x) + v(x)$, for functions $u(x)$ and $v(x)$, then $f'(x) = u'(x) + v'(x)$.