Lesson 511 Derivatives with e^x and the Natural Log Functions

Basic Natural Log Graphs f(x) = ln(x)

- The domain is (0, ∞)
- The range is (-∞, ∞)
- The function is continuous, increasing, and one-to-one on its entire domain.
- The graph of is concave downward on its entire domain.
- The graph passes through (1, 0)
- The graph has a vertical asymptote at x = 0.
- $\lim_{x \to 0^+} \ln(x) = -\infty$
- $\lim_{x \to \infty} \ln(x) = \infty$

Graph:

Useful rules:

The natural log is a log of base e (e = 2.718)

$\ln(ab) = \ln(a) + \ln(b)$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$\ln(a^b) = b \bullet \ln(a)$

$e^{lnx} = x$

$\ln(e^x) = x$

Basics of exponentials f(x) = e^x

- The domain is (-∞, ∞)
- The range is (0, ∞)
- The function is continuous, increasing, and one-to-one on its entire domain.
- The graph of is concave upward on its entire domain.
- The graph passes through (0, 1)
- The graph has a horizontal asymptote at y = 0.
- $\lim_{x \to -\infty} e^x = 0$
- $\lim_{x \to \infty} e^x = \infty$

Graph:

Useful Rules:

$e^a e^b = e^{a+b}$

$\frac{e^a}{e^b} = e^{a-b}$

In Exercises 7–10, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

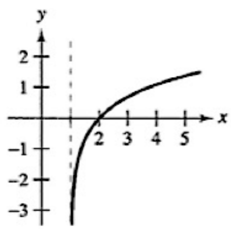
7. $f(x) = \ln x + 2$

8. $f(x) = -\ln x$

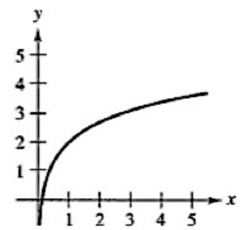
9. $f(x) = \ln(x - 1)$

10. $f(x) = -\ln(-x)$

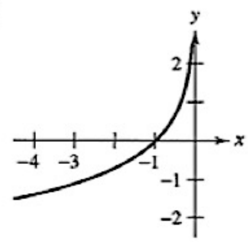
(a)



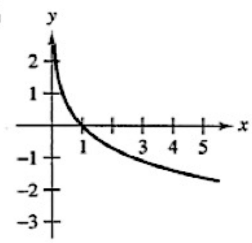
(b)



(c)



(d)



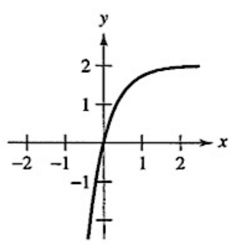
25. $y = Ce^{ax}$

26. $y = Ce^{-ax}$

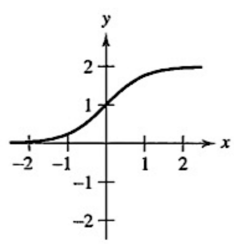
27. $y = C(1 - e^{-ax})$

28. $y = \dfrac{C}{1 + e^{-ax}}$

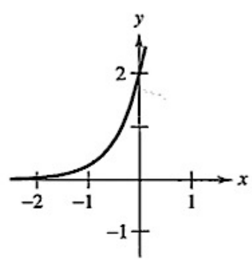
(a)



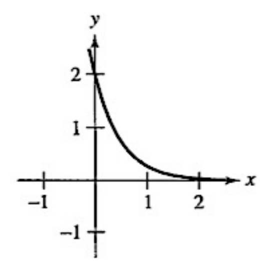
(b)



(c)



(d)



In Exercises 5–18, solve for x accurate to three decimal places.

5. $e^{\ln x} = 4$

7. $e^x = 12$

9. $9 - 2e^x = 7$

11. $50e^{-x} = 30$

13. $\ln x = 2$

15. $\ln(x - 3) = 2$

17. $\ln\sqrt{x + 2} = 1$

6. $e^{\ln 2x} = 12$

8. $4e^x = 83$

10. $-6 + 3e^x = 8$

12. $200e^{-4x} = 15$

14. $\ln x^2 = 10$

16. $\ln 4x = 1$

18. $\ln(x - 2)^2 = 12$

Derivatives

Function Name	f(x)	f'(x)
Natural Log	f(x) = ln(g(x))	$f'(x) = \frac{g'(x)}{g(x)}$

Function Name	f(x)	f'(x)
e^x	$f(x)=e^{g(x)}$	$f(x)=g'(x)\bullet e^{g(x)}$

Examples

A. y = ln(3x-1)

B. $y = \ln(4 - x^2)$

C. $y = e^{\sin(x)} - e^{7x}$

Differentiate each function with respect to *x*.

1) $y = \ln x^3$

2) $y = e^{2x^3}$

3) $y = \ln \ln 2x^4$

4) $y = \ln \ln 3x^3$

More Examples

5) $y = \cos \ln 4x^3$

6) $y = e^{e^{3x^2}}$

7) $y = e^{(4x^3+5)^2}$

8) $y = \ln 4x^2 \cdot (-x^3 - 4)$

9) $y = \ln \left(-\frac{4x^4}{x^3 - 3}\right)^5$

10) $y = \frac{e^{5x^4}}{e^{4x^2+3}}$

Problem Set 511 **Find the derivative for each function**

45. $g(x) = \ln x^2$

47. $y = (\ln x)^4$

49. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

51. $f(x) = \ln\left(\frac{x}{x^2 + 1}\right)$

53. $g(t) = \frac{\ln t}{t^2}$

55. $y = \ln(\ln x^2)$

57. $y = \ln\sqrt{\frac{x + 1}{x - 1}}$

59. $f(x) = \ln\left(\frac{\sqrt{4 + x^2}}{x}\right)$

39. $f(x) = e^{2x}$

41. $y = e^{-2x+x^2}$

43. $y = e^{\sqrt{x}}$

45. $g(t) = (e^{-t} + e^{t})^3$

47. $y = \ln(e^{x^2})$

49. $y = \ln(1 + e^{2x})$

51. $y = \frac{2}{e^x + e^{-x}}$

53. $y = x^2e^x - 2xe^x + 2e^x$

55. $f(x) = e^{-x}\ln x$

57. $y = e^x(\sin x + \cos x)$

Problem Set 512 **Find the derivative for each function**

46. $h(x) = \ln(2x^2 + 1)$

48. $y = x \ln x$

50. $y = \ln\sqrt{x^2 - 4}$

52. $f(x) = \ln\left(\frac{2x}{x + 3}\right)$

54. $h(t) = \frac{\ln t}{t}$

56. $y = \ln(\ln x)$

58. $y = \ln\sqrt[3]{\frac{x - 1}{x + 1}}$

60. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

40. $f(x) = e^{1-x}$

42. $y = e^{-x^2}$

44. $y = x^2e^{-x}$

46. $g(t) = e^{-3/t^2}$

48. $y = \ln\left(\frac{1 + e^x}{1 - e^x}\right)$

50. $y = \ln\frac{e^x + e^{-x}}{2}$

52. $y = \frac{e^x - e^{-x}}{2}$

54. $y = xe^x - e^x$

56. $f(x) = e^3 \ln x$

58. $y = \ln e^x$

Key

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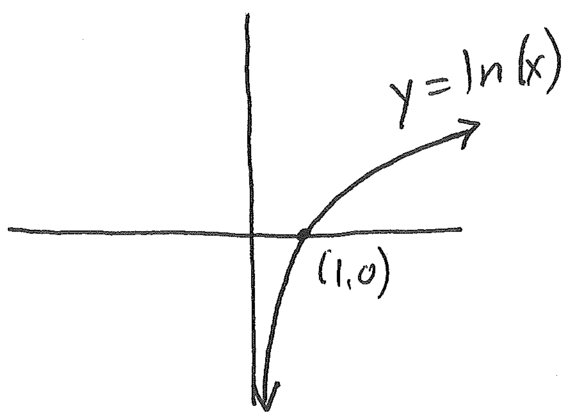
Graph:



Useful rules:

The natural log is a log of base

$\ln(ab) = \ln(a) + \ln(b)$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

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$e^{lnx} = x$

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Basics of exponentials $f(x) = e^x$

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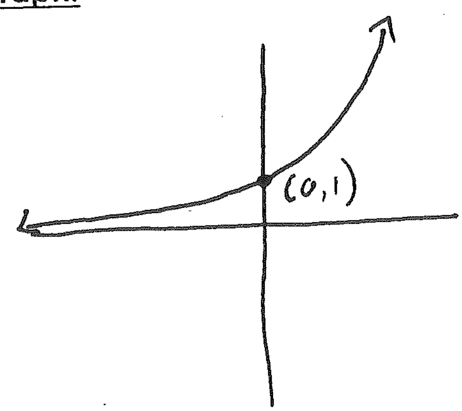


Useful Rules:

e $(e = 2.718)$

$e^a e^b = e^{a+b}$

$\frac{e^a}{e^b} = e^{a-b}$

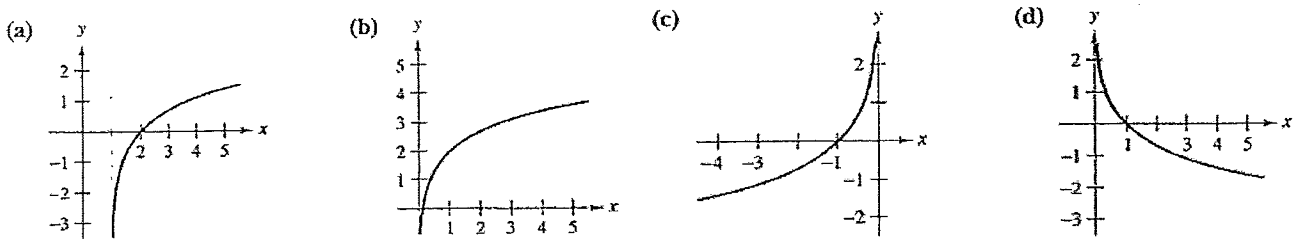
In Exercises 7–10, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

7. $f(x) = \ln x + 2$ B

8. $f(x) = -\ln x$ D

9. $f(x) = \ln(x - 1)$ A

10. $f(x) = -\ln(-x)$ C



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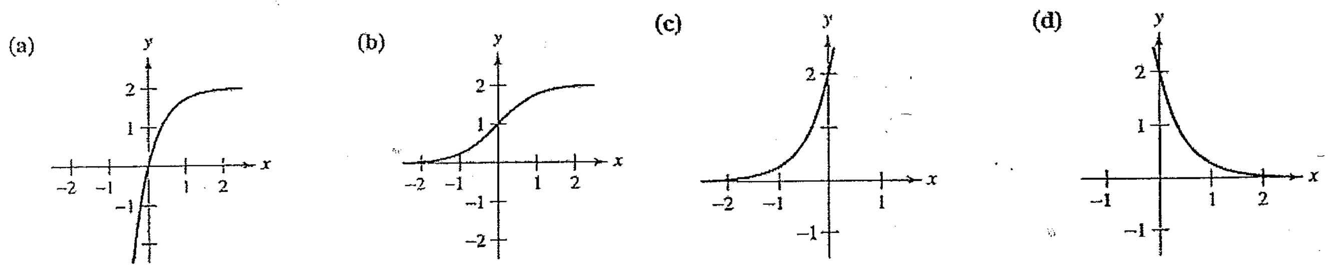

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11. $50e^{-x} = 30$ $e^{-x} = \frac{3}{5}$ $-x = \ln\left(\frac{3}{5}\right)$ $x = -\ln(3/5)$

13. $\ln x = 2$ $x = e^2$

15. $\ln(x - 3) = 2$ $x-3 = e^2$ $x = e^2+3$

17. $\ln\sqrt{x + 2} = 1$ $\sqrt{x+2} = e$ $x+2 = e^2$ $x = e^2-2$

6. $e^{\ln 2x} = 12$ $2x=12$ $x=6$

8. $4e^x = 83$ $e^x = \frac{83}{4}$ $x = \ln(83/4)$

10. $-6 + 3e^x = 8$ $e^x = \frac{14}{3}$ $x = \ln(14/3)$

12. $200e^{-4x} = 15$ $e^{-4x} = \frac{15}{200}$ $-4x = \ln(15/200)$ $x = -\frac{1}{4}\ln(15/200)$

14. $\ln x^2 = 10$

16. $\ln 4x = 1$ $4x = e$ $x = \frac{1}{4}e$

18. $\ln(x - 2)^2 = 12$

$(x-2)^2 = e^{12}$

$x-2 = e^6$

$x = e^6+2$

Derivatives

Function Name	f(x)	f'(x)
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Function Name	f(x)	f'(x)
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Examples

A. y = ln(3x-1)

$y' = \frac{3}{3x-1}$

B. $y = \ln(4 - x^2)$

$y' = \frac{-2x}{4-x^2}$

C. $y = e^{\sin(x)} - e^{7x}$

$y' = e^{\sin x} \cos x - e^{7x}(7)$

Differentiate each function with respect to x.

1) $y = \ln x^3$

$y' = \frac{3x^2}{x^3} = \frac{3}{x}$

2) $y = e^{2x^3}$

$y' = 6x^2 \cdot e^{2x^3}$

3) $y = \ln(\ln 2x^4)$

$y' = \frac{\frac{8x^3}{2x^4}}{\frac{\ln 2x^4}{1}} = \frac{8x^3}{2x^4} \cdot \frac{1}{\ln 2x^4}$

$= \frac{4}{x \cdot \ln 2x^4}$

4) $y = \ln \ln 3x^3$

$y' = \frac{\frac{9x^2}{3x^3}}{\frac{\ln(3x^3)}{1}}$

$\frac{3}{x} \cdot \frac{1}{\ln(3x^3)} = \frac{3}{x \ln(3x^3)}$

More Examples

5) $y = \cos(\ln 4x^3)$

$y' = -\sin(\ln 4x^3)\left(\frac{12x^2}{4x^3}\right)$

6) $y = e^{e^{3x^2}}$

$y' = e^{e^{3x^2}} \cdot e^{3x^2} \cdot 6x$

7) $y = e^{(4x^3+5)^2}$

$y' = e^{(4x^3+5)^2} \cdot 2(4x^3+5)12x^2$

8) $y = \ln 4x^2 \cdot (-x^3 - 4)$

$y' = \frac{8x}{4x^2}(-x^3-4) + \ln 4x^2(-3x^2)$

9) $y = \ln\left(-\frac{4x^4}{x^3-3}\right)^5$

$$y' = \frac{5\left(\frac{-4x^4}{x^3-3}\right)^4 \cdot \frac{(x^3-3)(-16x^3)+4x^4(3x^2)}{(x^3-3)^2}}{\left(\frac{-4x^4}{x^3-3}\right)^5}$$

10) $y = \frac{e^{5x^4}}{e^{4x^2+3}} = e^{5x^4-4x^2-3}$

$y' = e^{5x^4-4x^2-3} \cdot (20x^3-8x)$

$f(x) = \ln(g(x))$

$f'(x) = \frac{g'(x)}{g(x)}$

$f(x) = e^{g(x)}$

$f'(x) = e^{g(x)} \cdot g'(x)$

$\frac{x^6}{x^3} = x^{6-3}$

Problem Set 511 Find the derivative for each function

45. $g(x) = \ln x^2$ $\quad \frac{2x}{x^2} = \frac{2}{x}$

47. $y = (\ln x)^4$ $\quad 4(\ln x)^3\left(\frac{1}{x}\right)$

49. $y = \ln\left(x\sqrt{x^2-1}\right)$ $\longrightarrow$ $y' = \dfrac{1\cdot\sqrt{x^2-1} + x\frac{1}{2}(x^2-1)^{-1/2}(2x)}{x\sqrt{x^2-1}}$

51. $f(x) = \ln\left(\dfrac{x}{x^2+1}\right)$ $\quad f'(x) = \dfrac{\frac{(x^2+1)\cdot 1 - x(2x)}{(x^2+1)^2}}{\frac{x}{x^2+1}}$

53. $g(t) = \dfrac{\ln t}{t^2}$ $\quad \dfrac{t^2\left(\frac{1}{t}\right) - 2t\ln t}{t^4}$

55. $y = \ln(\ln x^2)$ $\quad y' = \dfrac{\frac{2x}{x^2}}{\ln(x^2)}$

57. $y = \ln\sqrt{\dfrac{x+1}{x-1}}$ $\quad y' = \dfrac{\frac{1}{2}\left(\frac{x+1}{x-1}\right)^{-1/2}\left(\frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2}\right)}{\sqrt{\frac{x+1}{x-1}}}$

59. $f(x) = \ln\left(\dfrac{\sqrt{4+x^2}}{x}\right)$ $\quad y' = \dfrac{\frac{x\frac{1}{2}(4+x^2)^{-1/2}(2x) - \sqrt{4+x^2}\cdot 1}{x^2}}{\frac{\sqrt{4+x^2}}{x}}$

39. $f(x) = e^{2x}$ $\quad f'(x) = 2e^{2x}$

41. $y = e^{-2x+x^2}$ $\quad y' = e^{-2x+x^2}(-2+2x)$

43. $y = e^{\sqrt{x}}$ $\quad y' = e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)$

45. $g(t) = (e^{-t} + e^{t})^3$ $\quad g'(t) = 3(e^{-t}+e^{t})^2(-e^{-t}+e^{t})$

47. $y = \ln(e^{x^2})$ $\quad y = x^2 \quad y' = 2x$

49. $y = \ln(1 + e^{2x})$ $\quad y' = \dfrac{2e^{2x}}{1+e^{2x}}$

51. $y = \dfrac{2}{e^x + e^{-x}}$ $\quad y = 2(e^x+e^{-x})^{-1} \quad y' = -2(e^x+e^{-x})^{-2}(e^x - e^{-x})$

53. $y = x^2e^x - 2xe^x + 2e^x$ $\quad y' = 2xe^x + x^2e^x - 2e^x - 2xe^x + 2e^x$

55. $f(x) = e^{-x}\ln x$ $\quad f'(x) = -e^{-x}\ln x + e^{-x}\left(\frac{1}{x}\right)$

57. $y = e^x(\sin x + \cos x)$

$y' = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

$y' = 2e^x\cos x$

Problem Set 512 Find the derivative for each function

46. $h(x) = \ln(2x^2 + 1)$ $h'(x) = \frac{4x}{2x^2+1}$

48. $y = x \ln x$ $y' = \ln x + \frac{x}{x}$ } $\ln x + 1$

50. $y = \ln\sqrt{x^2 - 4}$ $y' = \frac{\frac{1}{2}(x^2-4)^{-\frac{1}{2}}(2x)}{\sqrt{x^2-4}}$

52. $f(x) = \ln\left(\frac{2x}{x+3}\right)$ $f'(x) = \frac{\frac{(x+3)2-2x(1)}{(x+3)^2}}{\frac{2x}{x+3}}$

54. $h(t) = \frac{\ln t}{t}$ $h'(t) = \frac{t(\frac{1}{t}) - 1 \cdot \ln t}{t^2}$

56. $y = \ln(\ln x)$ $y' = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x \cdot \ln x}$

58. $y = \ln\sqrt[3]{\frac{x-1}{x+1}}$ $y' = \frac{\frac{1}{3}\left(\frac{x-1}{x+1}\right)^{-2/3} \cdot \left(\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}\right)}{\left(\frac{x-1}{x+1}\right)^{1/3}}$

60. $f(x) = \ln(x + \sqrt{4 + x^2})$ $f'(x) = \frac{1 + \frac{1}{2}(4+x^2)^{-\frac{1}{2}}(2x)}{x + \sqrt{4+x^2}}$

40. $f(x) = e^{1-x}$ $f'(x) = -e^{1-x}$

42. $y = e^{-x^2}$ $y' = -2xe^{-x^2}$

44. $y = x^2 e^{-x}$ $y' = 2x e^{-x} + -x^2 e^{-x}$

46. $g(t) = e^{-3/t^2}$ $g'(t) = e^{-3t^{-2}}(6t^{-3})$

48. $y = \ln\left(\frac{1+e^x}{1-e^x}\right)$ $y' = \frac{\frac{(1-e^x)(e^x)-(1+e^x)(-e^x)}{(1-e^x)^2}}{\frac{1+e^x}{1-e^x}}$

50. $y = \ln\frac{e^x + e^{-x}}{2}$ $y' = \frac{\frac{1}{2}e^x - \frac{1}{2}e^{-x}}{\frac{e^x+e^{-x}}{2}}$ ← $\frac{1}{2}e^x + \frac{1}{2}e^{-x}$

52. $y = \frac{e^x - e^{-x}}{2}$ $y = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ $y' = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

54. $y = xe^x - e^x$ $y' = 1e^x + xe^x - e^x = xe^x$

56. $f(x) = e^3 \ln x$ $f'(x) = \frac{e^3}{x}$ e^3 is a constant

58. $y = \ln e^x$
$y = x$ $y' = 1$