Calculus Lesson 453: U Substitution with complicated examples

Examples

$$\int (x-21)(x+7)^7 dx$$

$$\int x\sqrt{x+4}\, dx$$

$$\int x^2\sqrt{x-2}\, dx$$

$$\int \frac{\sec\left(\frac{1}{x^2}\right)tan\left(\frac{1}{x^2}\right)}{x^3}\, dx$$

Calculus Lesson 453: U Substitution with complicated examples

Examples

$\int (x-21)(x+7)^7 dx$

$u = x+7 \quad x = u-7$

$du = dx$

$\int (u-7-21)u^7 du = \int (u-28)u^7 du = \int (u^8 - 28u^7) du$

$\frac{1}{9}u^9 - \frac{28}{8}u^8 \rightarrow \frac{1}{9}(x+7)^9 - \frac{7}{2}(x+7)^8 + C$

$\int x\sqrt{x+4}\, dx$

$u = x+4$

$du = dx$

$x = u-4$

$\int (u-4)u^{1/2} du$

$\int (u^{3/2} - 4u^{1/2}) du$

$\frac{2}{5}u^{5/2} - \frac{4 \cdot 2}{1 \cdot 3}u^{3/2}$

$\frac{2}{5}(x+4)^{5/2} - \frac{8}{3}(x+4)^{3/2} + C$

$\int x^2\sqrt{x-2}\, dx$

$u = x-2$

$du = dx$

$x = u+2$

$\int (u+2)^2 u^{1/2} du = \int (u^2+4u+4)u^{1/2} du$

$= \int (u^{5/2} + 4u^{3/2} + 4u^{1/2}) du$

$= \frac{2}{7}u^{7/2} + \frac{4 \cdot 2}{1 \cdot 5}u^{5/2} + \frac{4 \cdot 2}{1 \cdot 3}u^{3/2}$

$= \frac{2}{7}(x-2)^{7/2} + \frac{8}{5}(x-2)^{5/2} + \frac{8}{3}(x-2)^{3/2} + C$

$\int \frac{\sec\left(\frac{1}{x^2}\right)\tan\left(\frac{1}{x^2}\right)}{x^3} dx$

$u = x^{-2}$

$du = -2x^{-3} dx$

$-\frac{1}{2} du = \frac{1}{x^3} dx$

$= -\frac{1}{2}\int \sec(u)\tan(u) du$

$= -\frac{1}{2}\sec(u)$

$= -\frac{1}{2}\sec\left(\frac{1}{x^2}\right) + C$

Revisiting Fundamental Theorem of Calculus part 2 using substitution

Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t dt$. Find $F'(x)$.

Solution

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin t dt$. Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$\begin{aligned} F'(x) &= \sin(u(x))\frac{du}{dx} \\ &= \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) \\ &= \frac{\sin\sqrt{x}}{2\sqrt{x}}. \end{aligned}$$

Let $F(x) = \int_x^{2x} t^3 dt$. Find $F'(x)$.

Solution

We have $F(x) = \int_x^{2x} t^3 dt$. Both limits of integration are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3 dt \\ &= \int_x^0 t^3 dt + \int_0^{2x} t^3 dt \\ &= -\int_0^x t^3 dt + \int_0^{2x} t^3 dt. \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3 dt\right] = -x^3.$$

Differentiating the second term, we first let $u(x) = 2x$. Then,

$$\begin{aligned} \frac{d}{dx}\left[\int_0^{2x} t^3 dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3 dt\right] \\ &= (u(x))^3 \frac{du}{dx} \\ &= (2x)^3 \cdot 2 \\ &= 16x^3. \end{aligned}$$

Thus,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3 dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3 dt\right] \\ &= -x^3 + 16x^3 \\ &= 15x^3. \end{aligned}$$

Problems

8) $\int x(4x-1)^4\,dx$

15) $\int x(4x+5)^3\,dx$

16) $\int 5x\sqrt{2x+3}\,dx$

271. $\int x(1-x)^{99}\,dx$

277. $\int \cos^2(\pi t)\sin(\pi t)dt$

278. $\int \sin^2 x\cos^3 x dx$ (*Hint*: $\sin^2 x+\cos^2 x=1$)

279. $\int t\sin(t^2)\cos(t^2)dt$

280. $\int t^2\cos^2(t^3)\sin(t^3)dt$

281. $\int \frac{x^2}{(x^3-3)^2}dx$

282. $\int \frac{x^3}{\sqrt{1-x^2}}dx$

Problems

(8) $\int x(4x-1)^4\,dx = \frac{1}{4}\int \left(\frac{1}{4}u+\frac{1}{4}\right)u^4\,du$

$u=4x-1 \quad x=\frac{1}{4}u+\frac{1}{4}$

$du=4dx$

$\frac{1}{4}du=dx$

$=\frac{1}{16}\int (u^5+u^4)\,du$

$=\frac{1}{16}\left[\frac{1}{6}u^6+\frac{1}{5}u^5\right]$

$=\frac{1}{72}(4x-1)^6+\frac{1}{60}(4x-1)^5+C$

15) $\int x(4x+5)^3\,dx$

(16) $\int 5x\sqrt{2x+3}\,dx = 5\cdot\frac{1}{2}\cdot\frac{1}{2}\int (u-3)u^{1/2}\,du$

$u=2x+3 \quad x=\frac{1}{2}(u-3)$

$du=2dx$

$\frac{1}{2}du=dx$

$=\frac{5}{4}\int \left(u^{3/2}-3u^{1/2}\right)du$

$=\frac{5}{4}\left(\frac{2}{5}u^{5/2}-\frac{3}{1}\cdot\frac{2}{3}u^{3/2}\right)$

$=\frac{1}{2}(2x+3)^{5/2}-\frac{5}{2}(2x+3)^{3/2}+C$

271. $\int x(1-x)^{99}\,dx$

277. $\int \cos^2(\pi t)\sin(\pi t)\,dt$

(278) $\int \sin^2 x\cos^3 x\,dx$ ~~(Hint: $\sin^2 x+\cos^2 x=1$)~~

$\cos^2 x = 1-\sin^2 x$

$u=\sin x$

$du=\cos x\,dx$

$\int \sin^2 x\cos^2 x\cos x\,dx$

$\int \sin^2 x(1-\sin^2 x)\cos x\,dx$

$\int u^2(1-u^2)\,du = \int (u^2-u^4)\,du$

$=\frac{1}{3}u^3-\frac{1}{5}u^5$

$=\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$

279. $\int t\sin(t^2)\cos(t^2)\,dt$

(280) $\int t^2\cos^2(t^3)\sin(t^3)\,dt$

$u=\cos(t^3)$

$du=-\sin(t^3)3t^2\,dt$

$-\frac{1}{3}du=\sin(t^3)t^2\,dt$

$-\frac{1}{3}\int u^2\,du \rightarrow \frac{-1}{3}\frac{1}{3}u^3 \rightarrow \frac{-1}{9}\left(\cos^3(t^3)\right)+C$

281. $\int \frac{x^2}{(x^3-3)^2}\,dx$

(282) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

$u=1-x^2 \quad x^2=1-u$

$du=-2x\,dx$

$-\frac{1}{2}du=x\,dx$

$\int \frac{x^2\,x\,dx}{\sqrt{1-x^2}} = \int u^{-1/2}(1-u)\left(-\frac{1}{2}du\right)$

$=\int \left(-\frac{1}{2}u^{-1/2}+\frac{1}{2}u^{1/2}\right)du$

$-(1-x^2)^{1/2}+\frac{1}{3}(1-x^2)^{3/2}+C$