Calculus Lesson 452: U Substitution with definite integrals

Definite Integral without changing the limits of integration

Example $\int_1^2 12x^3(3x^4+4)^4 dx$

1. Identify the inside function and set it equal to u

2. scratch work: find the derivative of the u equation in your scratch work.

Problem Solving: If all values in the original function cannot be replaced, consider:
1. **Multiply both sides by a constant in the scratch work**
2. **Redo the scratch work, but solve for x before differentiating.**

3. rewrite the original function using u and du

4. find the antiderivative

5. rewrite the antiderivative in terms of x

6. Apply the Fundamental Theorem of Calculus with the limits of integration.

Calculus Lesson 452: U Substitution with definite integrals

Definite Integral without changing the limits of integration

Example $$\int_1^2 12x^3(3x^4+4)^4\,dx$$

1. Identify the inside function and set it equal to u

$u = 3x^4 + 4$

2. scratch work: find the derivative of the u equation in your scratch work.

$du = 12x^3\,dx$

Problem Solving: If all values in the original function cannot be replaced, consider:
1. **Multiply both sides by a constant in the scratch work**
2. **Redo the scratch work, but solve for x before differentiating.**

3. rewrite the original function using u and du

$\int_1^2 12x^3(3x^4+4)^4\,dx$

$\int u^4\,du$

4. find the antiderivative

$\frac{1}{5}u^5$

5. rewrite the antiderivative in terms of x

$\frac{1}{5}(3x^4+4)^5 \Big|_1^2$

6. Apply the Fundamental Theorem of Calculus with the limits of integration.

$\frac{1}{5}(3(2)^4+4)^5 - \frac{1}{5}(3(1)^4+4)^5$

$76640806.4 - 3361.4$

$\boxed{76037445}$

Definite Integral with changing the limits of integration

Example $$\int_1^2 6x(x^2+1)^2 dx$$

1. Identify the inside function and set it equal to u

2. scratch work: find the derivative of the u equation in your scratch work.

3. use the equation for u to restate the upper and lower limits of integration.

4. rewrite the original function using u and du and the new limits of integration.

Problem Solving: If all values in the original function cannot be replaced, consider:

1. **Multiply both sides by a constant in the scratch work**
2. **Redo the scratch work, but solve for x before differentiating.**

5. find the antiderivative

6. substitute for u to write the antiderivative in terms of x

7. Apply the Fundamental Theorem of Calculus with the limits of integration.

Definite Integral with changing the limits of integration

Example $\int_1^2 6x(x^2+1)^2 dx$

1. Identify the inside function and set it equal to u

$u = x^2 + 1$

2. scratch work: find the derivative of the u equation in your scratch work.

$du = 2x\,dx$

$3du = 6x\,dx$

Problem Solving: If all values in the original function cannot be replaced, consider:

1. **Multiply both sides by a constant in the scratch work**
2. **Redo the scratch work, but solve for x before differentiating.**

* 3. use the equation for u to restate the upper and lower limits of integration. $u = x^2+1$

In x Upper = 2 $u = 2^2+1 = 5$ ← new upper

Lower = 1 $u = 1^2+1 = 2$ ← new lower

4. rewrite the original function using u and du and the new limits of integration.

$\int_1^2 (x^2+1)^2 6x\,dx = \int_2^5 u^2 (3du)$

5. find the antiderivative

$u^3 \Big|_2^5$

~~6. substitute for u to write the antiderivative in terms of x.~~

$5^3 - 2^3$

6. Apply the Fundamental Theorem of Calculus with the limits of integration.

$125 - 8 = \boxed{117}$

Problems

1) $\int_{-1}^{0} \frac{8x}{(4x^2+1)^2}\, dx;\ u = 4x^2 + 1$

2) $\int_{0}^{1} -12x^2(4x^3-1)^3\, dx;\ u = 4x^3 - 1$

3) $\int_{-1}^{2} 6x(x^2-1)^2\, dx$

4) $\int_{0}^{1} \frac{24x}{(4x^2+4)^2}\, dx$

5) $\int_{-3}^{0} -\frac{8x}{(2x^2+3)^2}\, dx$

6) $\int_{0}^{1} \frac{16x}{(4x^2+4)^2}\, dx$

7) $\int_{-1}^{0} 18x^2(3x^3+3)^2\, dx$

8) $\int_{0}^{1} -\frac{8x}{(4x^2+2)^2}\, dx$

292. $\int_0^1 x\sqrt{1-x^2}dx$

293. $\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

294. $\int_0^2 \frac{t}{\sqrt{5+t^2}}dt$

295. $\int_0^1 \frac{t}{\sqrt{1+t^3}}dt$

296. $\int_0^{\pi/4} \sec^2\theta\tan\theta d\theta$

Problems

1) $\displaystyle\int_{-1}^{0} \frac{8x}{(4x^2+1)^2}\,dx;\ u = 4x^2+1$

$u = 4x^2+1$

$du = 8x\,dx$

$\int u^{-2}\,du = -u^{-1} = \frac{-1}{u}$

$\left.\frac{-1}{4x^2+1}\right|_{-1}^{0} = \frac{-1}{4\cdot 0^2+1} - \frac{-1}{4(-1)^2+1}$

$-1 + \frac{1}{5} = \boxed{-\frac{4}{5}}$

2) $\displaystyle\int_{0}^{1} -12x^2(4x^3-1)^3\,dx;\ u = 4x^3-1$

$du = 12x^2dx$

$-\int u^3 du \rightarrow -\frac{1}{4}u^4 \rightarrow \left.-\frac{1}{4}(4x^3-1)^4\right|_0^1$

$-\frac{1}{4}(4(1)^3-1)^4 + \frac{1}{4}(4(0)^3-1)^4$

$\frac{-81}{4} + \frac{1}{4} = \frac{-80}{4} = \boxed{-20}$

3) $\displaystyle\int_{-1}^{2} 6x(x^2-1)^2\,dx$

$u = x^2-1$

$du = 2x\,dx$

$3du = 6x\,dx$

$3\int u^2 du$

u^3

$\left.(x^2-1)^3\right|_{-1}^{2}$

$(2^2-1)^3 - ((-1)^2-1)^3$

$27 - 0 = \boxed{27}$

4) $\displaystyle\int_{0}^{1} \frac{24x}{(4x^2+4)^2}\,dx = 3\int u^{-2}du$

$u = 4x^2+4$

$du = 8x\,dx$

$3du = 24x\,dx$

$-3u^{-1} = \left.\frac{-3}{4x^2+4}\right|_0^1$

$\frac{-3}{4(1)^2+4} - \frac{-3}{4(0)^2+4}$

$-\frac{3}{8} + \frac{3}{4} = -\frac{3}{8} + \frac{6}{8} = \boxed{\frac{3}{8}}$

5) $\displaystyle\int_{-3}^{0} -\frac{8x}{(2x^2+3)^2}\,dx = -2\int u^{-2}du$

$u = 2x^2+3$

$du = 4x\,dx$

$-2du = -8x\,dx$

$2u^{-1} = \frac{2}{u}$

$\left.\frac{2}{2x^2+3}\right|_{-3}^{0}$

$\frac{2}{2(0)^2+3} - \frac{2}{2(-3)^2+3} = \frac{2}{3} - \frac{2}{21} = \boxed{\frac{4}{7}}$

6) $\displaystyle\int_{0}^{1} \frac{16x}{(4x^2+4)^2}\,dx = 2\int u^{-2}du$

$u = 4x^2+4$

$du = 8x\,dx$

$2du = 16x\,dx$

$= -2u^{-1} = \frac{-2}{u}$

$= \left.\frac{-2}{4x^2+4}\right|_0^1$

$\frac{-2}{4(1)^2+4} - \frac{-2}{4(0)^2+4} = -\frac{2}{8} + \frac{2}{4} = \boxed{\frac{1}{4}}$

7) $\displaystyle\int_{-1}^{0} 18x^2(3x^3+3)^2\,dx$

$u = 3x^3+3$

$du = 9x^2dx$

$2du = 18x^2dx$

L: $u = 3(-1)^3+3 = 0$

U: $u = 3(0)^3+3 = 3$

$2\int_0^3 u^2 du$

$\left.\frac{2}{3}u^3\right|_0^3$

$\frac{2}{3}(3)^3 - \frac{2}{3}(0)^3$

$18 - 0$

$\boxed{18}$

8) $\displaystyle\int_{0}^{1} -\frac{8x}{(4x^2+2)^2}\,dx = -\int_2^6 u^{-2}du$

$u = 4x^2+2$

$du = 8x\,dx$

Low: $4(0)^2+2 = 2$

up: $4(1)^2+2 = 6$

$= \left.u^{-1}\right|_2^6$

$\frac{1}{6} - \frac{1}{2} = \boxed{-\frac{1}{3}}$

292. $\int_0^1 x\sqrt{1-x^2}\,dx$

$u = 1-x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

Up: $1-1^2=0$
Low: $1-0^2=1$

$-\frac{1}{2}\int_1^0 u^{1/2}\,du$

$\frac{1}{2}\int_0^1 u^{1/2}\,du$

$\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{1}{3}(1)^{3/2} - \frac{1}{3}(0)^{3/2}$
$= \boxed{1/3}$

293. $\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

294. $\int_0^2 \frac{t}{\sqrt{5+t^2}}dt$

$u = 5+t^2$
$du = 2t\,dt$
$\frac{1}{2}du = t\,dt$

Up: $5+2^2=9$
Low: $5+0^2=5$

$\frac{1}{2}\int_5^9 u^{-1/2}\,du$

$\frac{1}{2}\,\frac{2}{1}u^{1/2}\Big|_5^9$

$\sqrt{9}-\sqrt{5} = \boxed{3-\sqrt{5}}$

295. $\int_0^1 \frac{t}{\sqrt{1+t^3}}dt$

296. $\int_0^{\pi/4} \sec^2\theta\tan\theta\,d\theta$ $= \int_0^{\pi/4} \sec\theta\,\sec\theta\tan\theta\,d\theta$

$u = \sec\theta$
$du = \sec\theta\tan\theta\,d\theta$

$= \int_1^{\sqrt{2}} u\,du$

$= \frac{1}{2}u^2\Big|_1^{\sqrt{2}}$

Up: $\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Low: $\sec(0) = \frac{1}{\cos(0)} = 1$

$= \frac{1}{2}(\sqrt{2})^2 - \frac{1}{2}(1)^2$

$1-\frac{1}{2}$

$\boxed{\frac{1}{2}}$