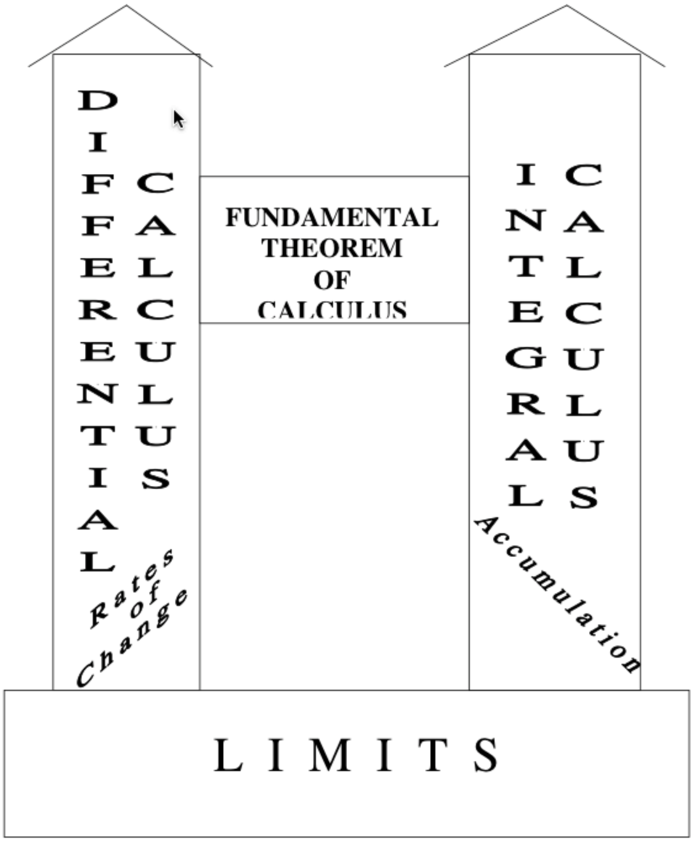
Calculus Lesson 444: Fundamental Theorem of Calculus, part 2



The Second Fundamental Theorem of Calculus

$$\boldsymbol{\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)}$$

Consider the Accumulation Function from a to x

$G(x) = \int_a^x f(t)dt$ $\qquad$ $G(x) = [F(t)]_a^x$ $\qquad$ G(x) = F(x) – F(a)

Notice that G(x) is found by integrating…

Now we will find the rate of change of G(x) with respect to x (d/dx)

$\frac{d}{dx}G(x) = \frac{d}{dx}F(x) - \frac{d}{dx}F(a)$ $\qquad$ which becomes $\quad$ g(x) = f(x) – 0

Notice that g(x) is found by differentiating…

Reminder - derivation and integration are inverse operations

Examples

$$\frac{d}{dx}\left[\int_3^x sin(t)dt\right]$$

Shortcut

Notice that the t is replaced by an x. This will be true with a constant on bottom and x on top.

Examples

$$\frac{d}{dx}\left[\int_3^x tan(t)dt\right]$$

$$\frac{d}{dx}\left[\int_3^x (e^t)dt\right]$$

$$\frac{d}{dx}\left[\int_3^x (sin(t) - t)dt\right]$$

Application

The rate of change of the AREA under f(x) at any point, x is equal to the height on f(x) at x.

Example

Find the area under the curve of $f(x) = x^2$ on [-2, 3]

Find the rate of change for the area at at x = 3

Complicated Limits of Integration

Alternate Process:

1. Define the inside as g
2. rewrite using g
3. Antiderivative: Perform middle step by writing G at the upper minus G at the lower
4. Derivative: switch each G back to g using the chain rule
5. Rewrite using the original function

Problem	Step 1 Define g	Step 2 rewrite	Step 3 ANTI	Step 4 DERIV	Step 5 rewrite
$\frac{d}{dx}\int_0^x 3t^2dt$	$g(t) = 3t^2$	$\frac{d}{dx}\int_0^x g(t)dt$	G(x) – G(0)	g(x)•1 – 0	$3x^2$
$\frac{d}{dx}\int_0^x (t^3+2t)dt$	$g(t) = t^3 + 2t$	$\frac{d}{dx}\int_0^x g(t)dt$	G(x) – G(0)	g(x)•1 – 0	$x^3 +2x$
$\frac{d}{dx}\int_0^{x+1} sintdt$	g(t) = sint	$\frac{d}{dx}\int_0^{x+1} g(t)dt$	G(x+1) – G(0)	g(x)•1 – 0	sin(x+1)
$\frac{d}{dx}\int_x^{x^3} ln(t)dt$	g(t)=ln t	$\frac{d}{dx}\int_x^{x^3} g(t)dt$	$G(x^3)$ – G(x)	$g(x^3)$•$3x^2$ – g(x)•1	$3x^2ln(x^3)$ – ln(x)

Tricky Examples:

$$\frac{d}{dx}\left[\int_2^{x^2} sin(t)dt\right] = sin(x^2)(2x) - sin(2) \bullet 0$$

$$\frac{d}{dx}\left[\int_x^{x^2} t^4dt\right] = (x^2)^4(2x) - (x)^4(1)$$

$$\frac{d}{dx}\left[\int_{2-x}^{x^2+x} cos(t)dt\right] = cos(x^2+x)(2x+1) - cos(2-x) \bullet (-1)$$

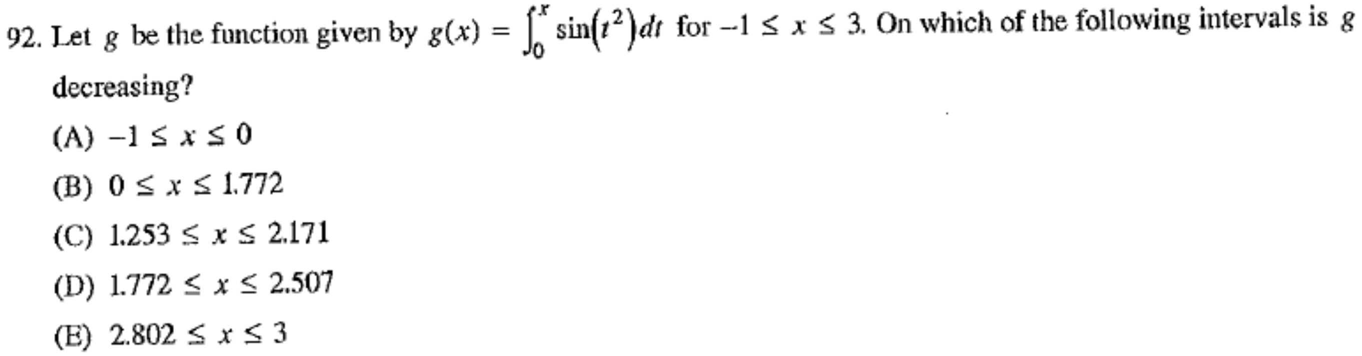
$$\frac{d}{dx}\left[\int_{sin(x)}^{cos(x)} (t^2)dt\right] = (cos(x))^2(-sin(x)) - (sin(x))^2(cos(x))$$

$$\frac{d}{dx}\left[\int_{sin(2x)}^{x^2} \sqrt{t}dt\right] = \sqrt{x^2}(2x) - \sqrt{sin(2x)}\ (cos(2x)(2)$$

Examples

$$\frac{d}{dx}\left[\int_{sin(x)}^{tan(x)} (t^3)dt\right]$$

$$\frac{d}{dx}\left[\int_{\sqrt{x}}^{x^2} \frac{1}{t}dt\right]$$



92. Let g be the function given by $g(x) = \int_0^x \sin(t^2)dt$ for $-1 \leq x \leq 3$. On which of the following intervals is g decreasing?

(A) $-1 \leq x \leq 0$

(B) $0 \leq x \leq 1.772$

(C) $1.253 \leq x \leq 2.171$

(D) $1.772 \leq x \leq 2.507$

(E) $2.802 \leq x \leq 3$

g(x) = $\int_3^x (t^5 - 2t)dt$ Find any max/mins on g(x)

Problems

148. $\frac{d}{dx}\int_1^x e^{-t^2}\,dt$

149. $\frac{d}{dx}\int_1^x e^{\cos t}\,dt$

150. $\frac{d}{dx}\int_3^x \sqrt{9-y^2}dy$

151. $\frac{d}{dx}\int_4^x \frac{ds}{\sqrt{16-s^2}}$

152. $\frac{d}{dx}\int_x^{2x} t dt$

153. $\frac{d}{dx}\int_0^{\sqrt{x}} t dt$

154. $\frac{d}{dx}\int_0^{\sin x} \sqrt{1-t^2}dt$

155. $\frac{d}{dx}\int_{\cos x}^1 \sqrt{1-t^2}dt$

156. $\frac{d}{dx}\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}dt$

157. $\frac{d}{dx}\int_1^{x^2} \frac{\sqrt{t}}{1+t}dt$

158. $\frac{d}{dx}\int_0^{\ln x} e^t\,dt$

159. $\frac{d}{dx}\int_1^{e^2} \ln u^2\,du$

Examples

$\frac{d}{dx}[\int_3^x sin(t)dt]$ $\frac{d}{dx}\left[-\cos t \Big|_3^x\right]$

$\frac{d}{dx}\left[-\cos x + \cos 3\right]$

$\boxed{\sin x}$

Shortcut

Notice that the t is replaced by an x. This will be true with a constant on bottom and x on top.

Examples

$\frac{d}{dx}\left[\int_3^x tan(t)dt\right]$ $= \tan x$

$\frac{d}{dx}\left[\int_3^x (e^t)dt\right]$ $= e^x$

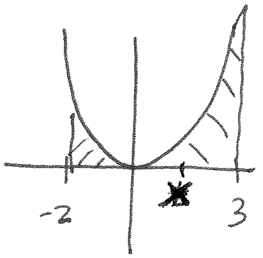
$\frac{d}{dx}\left[\int_3^x (sin(t) - t)dt\right]$ $= \sin x - x$

Application

The rate of change of the AREA under f(x) at any point, x is equal to the height on f(x) at x.

Example

Find the area under the curve of f(x) = x² on [-2, 3]



$= 11.6$

Find the rate of change for the area at at x = 3

rate is 9

~~$A = \int_{-2}^{t} x^2 dx$~~

$y = \int_{-2}^{x} t^2 dt$

$\frac{d}{dx} y = \frac{d}{dx} \int_{-2}^{x} t^2 dt = x^2$

Rate of Change of Area at X is x^2

Complicated Limits of Integration

Alternate Process:

1. Define the inside as g
2. rewrite using g
3. Antiderivative: Perform middle step by writing G at the upper minus G at the lower
4. Derivative: switch each G back to g using the chain rule
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Problem	Step 1 Define g	Step 2 rewrite	Step 3 ANTI	Step 4 DERIV	Step 5 rewrite
$\frac{d}{dx}\int_0^x 3t^2dt$	g(t) = 3t²	$\frac{d}{dx}\int_0^x g(t)dt$	G(x) – G(0)	g(x)•1 – 0	3x²
$\frac{d}{dx}\int_0^x (t^3+2t)dt$	$g(t) = t^3 + 2t$	$\frac{d}{dx}\int_0^x g(t)dt$	G(x) – G(0)	g(x)•1 – 0	x³ +2x
$\frac{d}{dx}\int_0^{x+1} sintdt$	g(t) = sint	$\frac{d}{dx}\int_0^{x+1} g(t)dt$	G(x+1) – G(0)	g(x)•1 – 0	sin(x+1)
$\frac{d}{dx}\int_x^{x^3} ln(t)dt$	g(t)=ln t	$\frac{d}{dx}\int_x^{x^3} g(t)dt$	G(x³) – G(x)	g(x³)•3x² – g(x)•1	3x²ln(x³) – ln(x)

Tricky Examples:

$\text{Anti}(x^2) - \text{Anti}(2) = \sin(x^2)2x - 0$

$$\frac{d}{dx}\left[\int_2^{x^2} sin(t)dt\right] = sin(x^2)(2x) - sin(2) \bullet 0$$

$$\frac{d}{dx}\left[\int_x^{x^2} t^4dt\right] = (x^2)^4(2x) - (x)^4(1)$$

$$\frac{d}{dx}\left[\int_{2-x}^{x^2+x} cos(t)dt\right] = cos(x^2+x)(2x+1) - cos(2-x) \bullet (-1)$$

$$\frac{d}{dx}\left[\int_{sin(x)}^{cos(x)} (t^2)dt\right] = (cos(x))^2(-sin(x)) - (sin(x))^2(cos(x))$$

$$\frac{d}{dx}\left[\int_{sin(2x)}^{x^2} \sqrt{t}dt\right] = \sqrt{x^2}(2x) - \sqrt{sin(2x)}\ (cos(2x)(2)$$

$\text{Anti}(x^2) - \text{Anti}(\sin(2x))$

$\sqrt{x^2}(2x) - \sqrt{\sin(2x)}\cos(2x)2$

Examples

$$\frac{d}{dx}\left[\int_{\sin(x)}^{\tan(x)} (t^3)\,dt\right] = \frac{d}{dx}\left[\text{Anti}(\tan x) - \text{Anti}(\sin x)\right]$$
$$= (\tan x)^3 \sec^2 x - (\sin x)^3(\cos x)$$

$$\frac{d}{dx}\left[\int_{\sqrt{x}}^{x^2} \frac{1}{t}\,dt\right] = \frac{d}{dx}\left[\text{Anti}(x^2) - \text{Anti}(\sqrt{x})\right]$$
$$= \frac{1}{x^2}(2x) - \frac{1}{\sqrt{x}}\left(\frac{1}{2}x^{-1/2}\right) = \frac{2x}{x^2} - \frac{1}{2\sqrt{x}\sqrt{x}} = \frac{2}{x} - \frac{1}{2x}$$

92. Let g be the function given by $g(x) = \int_0^x \sin(t^2)\,dt$ for $-1 \le x \le 3$. On which of the following intervals is g decreasing? $\longrightarrow$ when $g'(x)$ is negative

(A) $-1 \le x \le 0$

(B) $0 \le x \le 1.772$

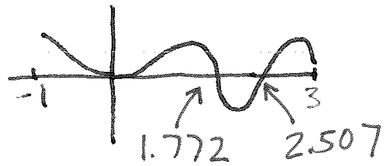
(C) $1.253 \le x \le 2.171$

[✓ (D)] $1.772 \le x \le 2.507$

(E) $2.802 \le x \le 3$

$$g'(x) = \frac{d}{dx}\int_0^x \sin(t^2)\,dt$$

$$g'(x) = \sin(x^2)$$



under the x-axis on $g'(x)$ means decreasing on $g(x)$

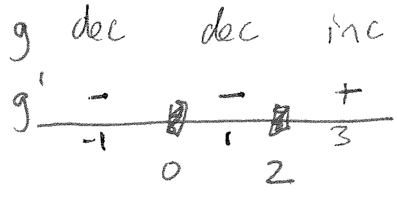
$g(x) = \int_3^x (t^5 - 2t)\,dt$ Find any max/mins on g(x)

$\downarrow$

critical points

$$g'(x) = \frac{d}{dx}\int_3^x (t^5 - 2t)\,dt$$
$$g'(x) = x^5 - 2x$$
$$0 = x^4(x-2)$$

critical points $x=0$ and $x=2$



$\boxed{x = 2 \text{ is a Min}}$

$$g(2) = \int_3^2 (t^5 - 2t)\,dt$$
$$= -\int_2^3 (t^5 - 2t)\,dt$$
$$= -\left[\frac{1}{6}t^6 - \frac{2}{2}t^2\right]_2^3$$
$$= -\left[\left(\frac{1}{6}(3)^6 - 3^2\right) - \left(\frac{1}{6}(2)^6 - 2^2\right)\right] = \boxed{-105.83}$$

Problems

148. $\frac{d}{dx}\int_1^x e^{-t^2}\,dt = e^{-x^2}$

149. $\frac{d}{dx}\int_1^x e^{\cos t}\,dt = e^{\cos x}$

150. $\frac{d}{dx}\int_3^x \sqrt{9-y^2}\,dy = \sqrt{9-x^2}$

151. $\frac{d}{dx}\int_4^x \frac{ds}{\sqrt{16-s^2}} = -\frac{1}{\sqrt{16-x^2}}$

152. $\frac{d}{dx}\int_x^{2x} t\,dt = \frac{d}{dx}\left[\text{Anti}(2x) - \text{Anti}(x)\right] = 2x(2) - x$

$= 4x - x = 3x$

153. $\frac{d}{dx}\int_0^{\sqrt{x}} t\,dt = \frac{d}{dx}\left[\text{Anti}(\sqrt{x}) - \text{Anti}(0)\right] = \sqrt{x}\left(\frac{1}{2}x^{-1/2}\right) - 0$

154. $\frac{d}{dx}\int_0^{\sin x} \sqrt{1-t^2}\,dt = \frac{d}{dx}\left[\text{Anti}(\sin x) - \text{Anti}(0)\right] = \sqrt{1-(\sin x)^2}\cos x - 0$

155. $\frac{d}{dx}\int_{\cos x}^1 \sqrt{1-t^2}\,dt = \frac{d}{dx}\left[\text{Anti}(1) - \text{Anti}(\cos x)\right] = 0 - \sqrt{1-(\cos x)^2}(-\sin x)$

156. $\frac{d}{dx}\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt = \frac{d}{dx}\left[\text{Anti}(\sqrt{x}) - \text{Anti}(1)\right] = \frac{(\sqrt{x})^2}{1+(\sqrt{x})^4} - 0 = \frac{x}{1+x^2}$

157. $\frac{d}{dx}\int_1^{x^2} \frac{\sqrt{t}}{1+t}\,dt = \frac{d}{dx}\left[\text{Anti}(x^2) - \text{Anti}(1)\right] = \frac{\sqrt{x^2}}{1+x^2}(2x) - 0 = \frac{2x^2}{1+x^2}$

158. $\frac{d}{dx}\int_0^{\ln x} e^t\,dt$

159. $\frac{d}{dx}\int_1^{e^2} \ln u^2\,du$