Calculus Lesson 442: Fundamental Theorem of Calculus, bounded Average Value of a Function

Investigation 1

Part 1 - What is the area under $f(x) = x^2 + 1$ from $x = 0$ to $x = 3$? Sketch the graph.

Part 2 - If we wanted a rectangle with the same area whose base sat on the x-axis from $x = 0$ to $x = 3$, then how tall is the rectangle? Sketch the graph.

Investigation 2

Part 1 - What is the area under $f(x) = \sin(x) + 3$ from $x = 0$ to $x = 1.5$□? Sketch the graph.

Part 2 - If we wanted a rectangle with the same area whose base sat on the x-axis from $x = 0$ to $x = 1.5$□, then how tall is the rectangle? Sketch the graph.

Summary

Average Value of a Function

$$\frac{1}{b-a}\int_{a}^{b} f(x)dx$$

Details

Area is measured in square units (Ex. square feet).

An integral finds the area from x = a to x = b.

Each area we find can be thought of as a rectangle with width (x axis) and height (y axis).

The width is the difference between a and b.

The area = width • height So height = area / width or height = area / (b – a)

Example

Find the average value of the function f(x) = x(x – 5) from x = 0 to x = 5

Application

The velocity of a car is described by $v(t) = -t^2 + 4t + 4$ units: m/s.
Find the average velocity from time t=1 to t = 3.
Calculate by hand, check on the calculator, and draw both curves.

Bounded Example

Find the area bounded by the graphs of $y = 2 + \sqrt{x}$ y = 0 x = 0 and x = 4

Problems

Evaluate the Integrals, then find the average value of the function on the interval.

179. $\int_1^4 \frac{1}{2\sqrt{x}}dx$

180. $\int_1^4 \frac{2-\sqrt{t}}{t^2}dt$

181. $\int_1^{16} \frac{dt}{t^{1/4}}$

182. $\int_0^{2\pi} \cos\theta d\theta$

183. $\int_0^{\pi/2} \sin\theta d\theta$

Find the area of the region bounded by $y = x^3 - x + 4$, $y = 0$, $x = -1$, and $x = 2$

Evaluate the Integrals

184. $\int_0^{\pi/4} \sec^2\theta d\theta$

185. $\int_0^{\pi/4} \sec\theta\tan\theta$

186. $\int_{\pi/3}^{\pi/4} \csc\theta\cot\theta d\theta$

187. $\int_{\pi/4}^{\pi/2} \csc^2\theta d\theta$

188. $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right)dt$

189. $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right)dt$

198. Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75\cos\left(\frac{\pi t}{6}\right) + 12.25,$ with t given in months and $t = 0$ corresponding to the winter solstice.

a. What is the average number of daylight hours in a year?

b. At which times t_1 and $t_2,$ where $0 \leq t_1 < t_2 < 12,$ do the number of daylight hours equal the average number?

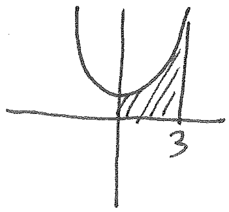
c. Write an integral that expresses the total number of daylight hours in Seattle between t_1 and $t_2.$

d. Compute the mean hours of daylight in Seattle between t_1 and $t_2,$ where $0 \leq t_1 < t_2 < 12,$ and then between t_2 and $t_1,$ and show that the average of the two is equal to the average day length.

Calculus Lesson 442: Fundamental Theorem of Calculus, bounded Average Value of a Function

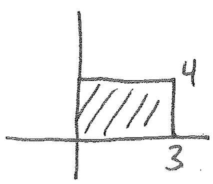
Investigation 1

Part 1 - What is the area under $f(x) = x^2 + 1$ from $x = 0$ to $x = 3$? Sketch the graph.



$$\int_0^3 (x^2+1)\,dx = 12$$

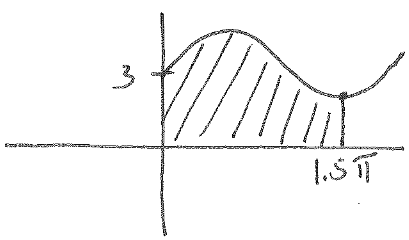
Part 2 - If we wanted a rectangle with the same area whose base sat on the x-axis from $x = 0$ to $x = 3$, then how tall is the rectangle? Sketch the graph.



Area = 12
Height = 4

Investigation 2

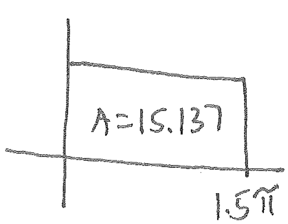
Part 1 - What is the area under $f(x) = \sin(x) + 3$ from $x = 0$ to $x = 1.5\pi$? Sketch the graph.



$$\int_0^{1.5\pi} (\sin x + 3)\,dx$$

15.137

Part 2 - If we wanted a rectangle with the same area whose base sat on the x-axis from $x = 0$ to $x = 1.5\pi$, then how tall is the rectangle? Sketch the graph.



$$\text{Height} = \frac{\text{Area}}{\text{width}}$$

$$\text{Height} = \frac{15.137}{1.5\pi} = 3.212$$

Summary

$$\text{Ave Value} = \frac{\text{Area}}{\text{width}} = \frac{\int_a^b (f(x))\,dx}{b-a}$$

Average Value of a Function

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Details

Area is measured in square units (Ex. square feet).

An integral finds the area from x = a to x = b.

Each area we find can be thought of as a rectangle with width (x axis) and height (y axis).

The width is the difference between a and b.

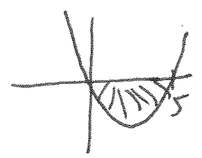
The area = width • height So height = area / width or height = area / (b – a)

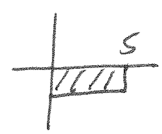
Example

Find the average value of the function f(x) = x(x – 5) from x = 0 to x = 5

$$\frac{1}{5}\int_0^5 x(x-5)dx = \frac{1}{5}\left(-20.8\overline{3}\right) = -4.1\overline{6}$$

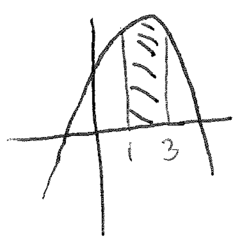
Height





Application

The velocity of a car is described by v(x) = -x^2 + 4x + 4 units: m/s.
Find the average velocity from time x=1 to x = 3.
Calculate by hand, check on the calculator, and draw both curves.



$$\int_1^3(-x^2+4x+4)dx = 15.33\overline{3}\ m$$

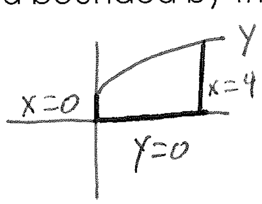
$$\text{Ave velocity} = \frac{15.\overline{3}}{2} = 7.\overline{6}\ m/s$$

$$\int_a^b v(t)dt = s(b) - s(a)$$

Net Displacement

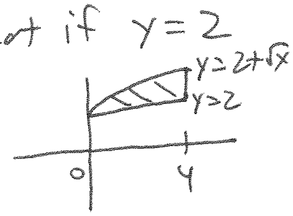
Bounded Example

Find the area bounded by the graphs of y = 2 + √x y = 0 and x = 4 + x = 0



$$\int_0^4(2+\sqrt{x})dx = 13.\overline{3}$$

what if y = 2



$$\int_0^4(2-\sqrt{x})dx - \int_0^4 2dx$$

$$13.\overline{3} - 8 = 5.\overline{3}$$

Problems

Evaluate the Integrals, then find the average value of the function on the interval.

179. $\int_1^4 \frac{1}{2\sqrt{x}}dx$ — $\frac{1}{2}x^{-1/2}$ — $\frac{\frac{1}{2}}{\frac{1}{2}}x^{1/2}\Big|_1^4 = \sqrt{4}-\sqrt{1} = \boxed{1}$ Ave $= \frac{1}{3}\cdot 1 = \boxed{\frac{1}{3}}$

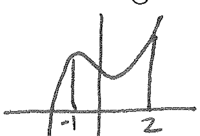
180. $\int_1^4 \frac{2-\sqrt{t}}{t^2}dt$ $= \int_1^4 (2t^{-2} - t^{-3/2})dt = -2t^{-1} + 2t^{-1/2}\Big|_1^4 = \left(\frac{-2}{4} + \frac{2}{\sqrt{4}}\right) - \left(\frac{-2}{1} + \frac{2}{\sqrt{1}}\right)$ $\frac{1}{2} - 0 = \boxed{\frac{1}{2}}$ Ave $\frac{1}{3}\cdot\frac{1}{2}$

181. $\int_1^{16} \frac{dt}{t^{1/4}}$ — $t^{-1/4}dt$ — $\frac{4}{3}t^{3/4}\Big|_1^{16} = \frac{4}{3}(16)^{3/4} - \frac{4}{3}(1)^{3/4} = \boxed{9.\overline{3}}$ Ave $= \frac{1}{15}\cdot\frac{28}{3} = \boxed{\frac{2}{4}}$

182. $\int_0^{2\pi} \cos\theta d\theta$ — $\sin\theta\Big|_0^{2\pi} = \sin(2\pi) - \sin(0) = \boxed{0}$ Ave $= \frac{1}{2\pi}\cdot 0 = \boxed{0}$

183. $\int_0^{\pi/2} \sin\theta d\theta$ — $-\cos\theta\Big|_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = \boxed{1}$ Ave $= \frac{1}{\frac{\pi}{2}}\cdot 1 = \boxed{\frac{2}{\pi}}$

Find the area of the region bounded by y = x³ – x + 4, y = 0, x = -1, and x = 2



$\int_{-1}^{2}(x^3 - x + 4)dx = \boxed{14.25}$

Evaluate the Integrals

184. $\int_0^{\pi/4} \sec^2\theta d\theta$ $= \tan\theta\Big|_0^{\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) - \tan(0) = \boxed{1}$

185. $\int_0^{\pi/4} \sec\theta\tan\theta$ $= \sec\theta\Big|_0^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec(0) = \boxed{.414}$ or $\frac{2}{\sqrt{2}} - 1 = \boxed{\sqrt{2}-1}$

186. $\int_{\pi/3}^{\pi/4} \csc\theta\cot\theta d\theta$ $= -\csc\theta\Big|_{\frac{\pi}{3}}^{\frac{\pi}{4}} = +\csc(\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \csc\left(\frac{\pi}{3}\right) - \csc\left(\frac{\pi}{4}\right) = \boxed{-.260}$ or $\frac{2}{\sqrt{3}} - \frac{2}{\sqrt{2}} = \boxed{\frac{2}{\sqrt{3}} - \dots}$

187. $\int_{\pi/4}^{\pi/2} \csc^2\theta d\theta$ $= -\cot\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right) = \boxed{1}$

undef, but $\lim_{x\to\frac{\pi}{2}} = 0 + 1$

188. $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right)dt$ — $t^{-2} - t^{-3}$ — $= \left[-t^{-1} + \frac{1}{2}t^{-2}\right]\Big|_1^2 = \left(\frac{-1}{2} + \frac{1}{2\cdot 2^2}\right) - \left(\frac{-1}{1} + \frac{1}{2\cdot 1^2}\right) = \frac{-3}{8} - \frac{-1}{2} = \boxed{\frac{1}{8}}$

189. $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right)dt$ — $t^{-2} - t^{-3}$ — $= \left[-t^{-1} + \frac{1}{2}t^{-2}\right]\Big|_{-2}^{-1} = \boxed{\frac{7}{8}}$

Same setup as 188

* Calculator

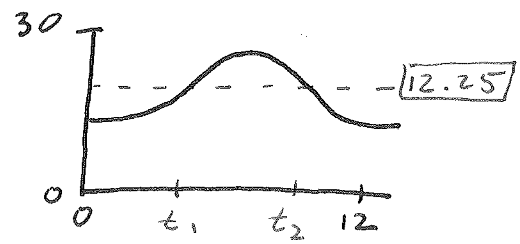
198. Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75\cos\left(\frac{\pi t}{6}\right) + 12.25$, with t given in months and $t = 0$ corresponding to the winter solstice.

a. What is the average number of daylight hours in a year?

b. At which times t_1 and t_2, where $0 \le t_1 < t_2 < 12$, do the number of daylight hours equal the average number?

c. Write an integral that expresses the total number of daylight hours in Seattle between t_1 and t_2.

d. Compute the mean hours of daylight in Seattle between t_1 and t_2, where $0 \le t_1 < t_2 < 12$, and then between t_2 and t_1, and show that the average of the two is equal to the average day length.



a) $\frac{1}{12}\int_0^{12}\left(-3.75\cos\left(\frac{\pi t}{6}\right) + 12.25\right)dt$

$\frac{1}{12}(147) = \boxed{12.25}$

b) $t_1 = 3 \qquad t_2 = 9$

c) $\int_3^9 \left(-3.75\cos\left(\frac{\pi t}{6}\right) + 12.25\right)dt$

d) $\frac{1}{6}\int_3^9\left(-3.75\cos\left(\frac{\pi t}{6}\right) + 12.25\right)dt = \frac{1}{6}(87.824) = 14.637$

$\frac{1}{6}\int_9^3 f(t)dt = \frac{1}{6}\int_9^{15} f(t)dt = \frac{1}{6}\left[\int_9^{12} f(t)dt + \int_{12}^{15} f(t)dt\right]$

$= \frac{1}{6}\left[\int_9^{12} f(t)dt + \int_0^3 f(t)dt\right]$

Think in years

Month 3 = month 15 (+12 months)

$= \frac{1}{6}\left[29.588 + 29.588\right] = 9.863$

Average $\frac{1}{2}(14.637 + 9.863) = \boxed{12.25}$

In Exercises 27–32, evaluate the definite integral of the trigonometric function. Use a graphing utility to verify your result.

27. $\int_0^{\pi} (1 + \sin x)\, dx$

28. $\int_0^{\pi/4} \frac{1 - \sin^2\theta}{\cos^2\theta}\, d\theta$

29. $\int_{-\pi/6}^{\pi/6} \sec^2 x\, dx$

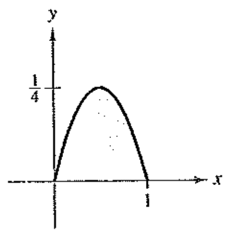
30. $\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)\, dx$

31. $\int_{-\pi/3}^{\pi/3} 4 \sec\theta \tan\theta\, d\theta$

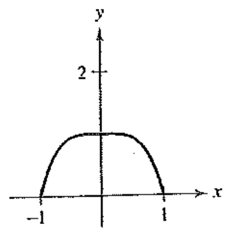
32. $\int_{-\pi/2}^{\pi/2} (2t + \cos t)\, dt$

In Exercises 35–40, determine the area of the indicated region.

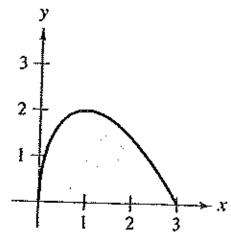
35. $y = x - x^2$



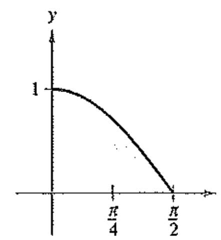
36. $y = 1 - x^4$



37. $y = (3 - x)\sqrt{x}$



39. $y = \cos x$



In Exercises 41–44, find the area of the region bounded by the graphs of the equations.

41. $y = 3x^2 + 1, \quad x = 0, \quad x = 2, \quad y = 0$

42. $y = 1 + \sqrt[3]{x} \quad x = 0, \quad x = 8, \quad y = 0$

In Exercises 49–52, find the average value of the function over the interval and all values of x in the interval for which the function equals its average value.

Function	*Interval*
49. $f(x) = 4 - x^2$	$[-2, 2]$
51. $f(x) = \sin x$	$[0, \pi]$
50. $f(x) = \frac{4(x^2 + 1)}{x^2}$	$[1, 3]$
52. $f(x) = \cos x$	$[0, \pi/2]$

Calculus Worksheet 4-4B Name Key Pd____

In Exercises 27–32, evaluate the definite integral of the trigonometric function. Use a graphing utility to verify your result.

N.C. **27.** $\int_0^{\pi} (1 + \sin x)\,dx$ $\left[x - \cos x\right]_0^{\pi} = [\pi - \cos\pi] - [0 - \cos 0]$ $\pi - (-1) + 1 = 2+\pi$

C **28.** $\int_0^{\pi/4} \frac{1 - \sin^2\theta}{\cos^2\theta}\,d\theta$ .785

C **29.** $\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx$ $: \int \left(\frac{1}{\cos x}\right)^2 = 1.154$

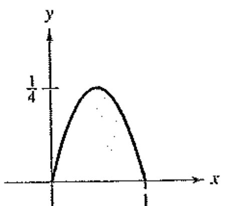
C **30.** $\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)\,dx$ $\int 2 - 1/(\sin(x)^2) = .570$

C **31.** $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta$ $4(1/\cos\theta)\tan\theta = 0$

C **32.** $\int_{-\pi/2}^{\pi/2} (2t + \cos t)\,dt$ = 2

In Exercises 35–40, determine the area of the indicated region.

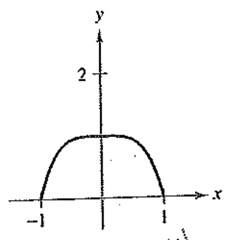
35. $y = x - x^2$



$\int_0^1 (x - x^2)dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$

$\left[\frac{1}{2} - \frac{1}{3}\right] - [0 - 0] = \frac{1}{6}$

36. $y = 1 - x^4$

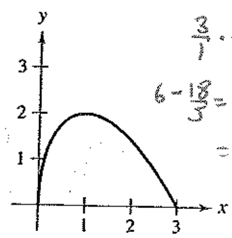


$\int_{-1}^{1}(1 - x^4) = \left[x - \frac{1}{5}x^5\right]_{-1}^{1}$

$\left[1 - \frac{1}{5}\right] - \left[-1 - \frac{1}{5}(-1)\right]$

$\frac{4}{5} - \left(-\frac{4}{5}\right) = \frac{8}{5}$

37. $y = (3 - x)\sqrt{x}$



$\frac{3}{1}\cdot\frac{2}{3} = 2$

$6 - \frac{18}{5} = \frac{30}{5} - \frac{18}{5} = \frac{12}{5}$

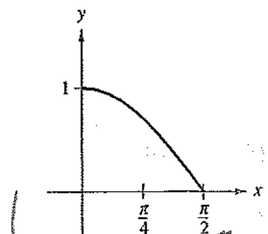
$\int_0^3 (3x^{1/2} - x^{3/2})dx = \left[\frac{3}{\frac{3}{2}}x^{3/2} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^3$

$\left[2(3)^{3/2} - \frac{2}{5}(3)^{\frac{5}{2}}\right] - [0 - 0]$

$2\cdot 3\sqrt{3} - \frac{2}{5}\cdot 9\sqrt{3}$

$6\sqrt{3} - \frac{18}{5}\sqrt{3} = \frac{12}{5}\sqrt{3}$

39. $y = \cos x$



$\int_0^{\frac{\pi}{2}} \cos x\,dx$

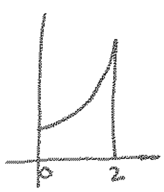
$[\sin x]_0^{\pi/2}$

$\sin\frac{\pi}{2} - \sin 0$

$1 - 0 = 1$

In Exercises 41–44, find the area of the region bounded by the graphs of the equations.

41. $y = 3x^2 + 1,\quad x = 0,\quad x = 2,\quad y = 0$

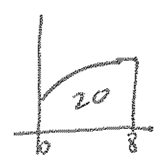


$\int_0^2 (3x^2 + 1)dx = \left[\frac{3}{3}x^3 + x\right]_0^2$

$[2^3 + 2] - [0^3 + 0]$

$8 + 2 = 10$

C **42.** $y = 1 + \sqrt[3]{x},\quad x = 0,\quad x = 8,\quad y = 0$



In Exercises 49–52, find the average value of the function over the interval and all values of x in the interval for which the function equals its average value.

Function	*Interval*
49. $f(x) = 4 - x^2$	$[-2, 2]$
C **50.** $f(x) = \frac{4(x^2 + 1)}{x^2}$	$[1, 3]$
51. $f(x) = \sin x$	$[0, \pi]$
C **52.** $f(x) = \cos x$	$[0, \pi/2]$

49. $\frac{24}{3} - \frac{8}{3} = \frac{16}{3}$ $\quad -\frac{24}{3} + \frac{8}{3} = -\frac{16}{3}$

$\frac{1}{2-(-2)}\int_{-2}^{2}(4 - x^2)dx = \frac{1}{4}\left[4x - \frac{1}{3}x^3\right]_{-2}^{2}$

$\frac{1}{4}\left[4(2) - \frac{1}{3}(2)^3\right] - \frac{1}{4}\left[4(-2) - \frac{1}{3}(-2)^3\right] = \frac{1}{4}\left[\frac{16}{3}\right] - \frac{1}{4}\left[\frac{-16}{3}\right] = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

50. $\frac{1}{3-1}\int_1^3 f(x)dx = \frac{1}{2}[10.\overline{6}] = 5.\overline{3}$

51. $\frac{1}{\pi - 0}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$

$\frac{1}{\pi}[-\cos\pi] - \frac{1}{\pi}[-\cos 0] = \frac{1}{\pi}(1) - \frac{1}{\pi}(-1)$

$\frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$

52. $\frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}}\cos x\,dx = \frac{1}{\frac{\pi}{2}}[1] = \frac{2}{\pi}$